

Callan-Symanzik method for m -axial Lifshitz points

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Abstract

We introduce the Callan-Symanzik method in the description of anisotropic as well as isotropic Lifshitz critical behaviors. Renormalized perturbation theories are defined by normalization conditions with nonvanishing masses and at zero external momenta. The orthogonal approximation is employed to obtain the critical indices η_{L2} , ν_{L2} , η_{L4} and ν_{L4} diagrammatically at least up to two-loop order in the anisotropic criticalities. This approximation is also utilized to compute the exponents η_{L4} and ν_{L4} in the isotropic case. Furthermore, we compute those exponents exactly for the isotropic behaviors at the same loop order. The results obtained for all exponents are in perfect agreement with those previously derived in the massless theories renormalized at nonzero external momenta.

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I. INTRODUCTION

Renormalized perturbative expansion of a $\lambda\phi^4$ field theory is the natural mathematical setting in statistical mechanics to determine the critical properties taking place in second order phase transitions. For ordinary critical systems, the theory is defined on a d -dimensional Euclidean space and has quadratic kinetic terms in momenta space, as usual in quantum field theory. These criticalities can be generalized by adding certain combinations of m quartic derivatives of the field to the kinetic term of the bare Lagrangian density. In fact, when the m coefficients of the usual quadratic derivatives of the field vanish, the Lagrangian density describes the so-called m -axial Lifshitz critical behaviors[1, 2, 3]. Physically, the bare Lagrangian describes an Ising model with ferromagnetic exchange forces between nearest neighbor spins as well as antiferromagnetic interactions between second neighbors along m space directions. The competition among ferro- and antiferromagnetic couplings of the spins in the model induces the vanishing of the m terms in the quadratic derivatives of the field keeping, however, the quartic derivatives along the competing directions. The m higher derivative terms will generate space anisotropy whenever $m < d$, with two inequivalent subspaces, whereas if $m = d$ there is only one isotropic space.

From the quantum field-theoretic side, interest in the model stems from the fact that it has a natural connection with Lorentz symmetry breaking for fields propagating in specific backgrounds in the long-distance limit. Indeed, a recent proposal postulates the existence of a physical fluid originating from a scalar field whose vacuum expectation value changes with a constant velocity [4]. This background breaks the time diffeomorphism symmetry generating a type of Goldstone boson for this breaking of Lorentz invariance, the ghost condensate (analogous to the Goldstone bosons which arise from spontaneous breaking of internal/gauge symmetries). It mimics a type of dark energy, whose low-energy effective Lagrangian in flat spacetime has kinetic terms with quadratic time derivatives and quartic space derivatives. The ghost condensate mixes with gravity in a kind of Higgs mechanism giving rise to a nontrivial modification of gravity in the infrared [5]. The competing interactions between attractive and repulsive components of gravity at large scales manifest themselves in the absence of quadratic space derivatives of the ghost condensate in the kinetic Lagrangian implying that this flat spacetime is not the conventional Minkowski space. After a Wick rotation in time for this quantum field theory, the kinetic Lagrangian for the ghost condensate

(π) can be identified with that for the order parameter field (magnetization ϕ) in the m -axial Lifshitz critical behavior for $m = 3$. Therefore, figuring out the perturbative structure of Lifshitz points is worthwhile in order to gain a better comprehension of this kind of Lorentz breaking quantum field theory as well as the large distance effects on the gravity sector in this model.

Renormalization group and ϵ -expansion ideas ($\epsilon = 4 - d$) were originally introduced to provide the perturbative determination of critical exponents using diagrammatic methods in momenta space [6]. A reformulation of this method was introduced to compute the critical exponents in a massless theory renormalized at nonzero external momenta[7, 8]. The associated renormalization group equation along with further developments in evaluating Feynman diagrams have been adapted to study critical exponents of the Lifshitz type when more than one characteristic length scale are present [9]. Critical exponents of Lifshitz points have previously been obtained applying these ideas in massless theories renormalized at nonzero external momenta[10, 11].

An alternative to the above methods is to formulate the problem in a massive theory. This permits to comprehend under what conditions the scale invariance of the solution to the renormalization group equation for the renormalized 1PI vertex parts is guaranteed. In addition, this formulation is convenient to treat renormalization issues of arbitrary quantum field theories and it is a healthy test for the universality hypothesis, which guarantees that universal quantities (e.g., the critical exponents) do not depend on the scheme of renormalization. For ordinary critical systems, the asymptotic scale invariance of the theory in $d = 4 - \epsilon$ was established more rigorously after the first developments Ref.[6] through the use of Callan-Symanzik equations [12, 13] in the explicit computation of the critical exponents for a massive theory[14]. However, the same approach has not been applied for investigating more general critical systems so far. In fact we would like to know if such a method could be adapted to critical systems exhibiting more than one characteristic length scale, such as the m -fold Lifshitz critical behavior. It is the purpose of the present work to show that such a mission can be accomplished, following closely the arguments put forth in [14]. Here, we derive the critical exponents η and ν in the massive case by introducing an appropriate Callan-Symanzik formalism with two independent mass scales in the bare Lagrangian corresponding to the correlation lengths ξ_{L2} and ξ_{L4} associated to the anisotropic critical behaviors. The massive theories are renormalized at zero external momenta. A

similar treatment will be shown to be valid for the isotropic behaviors with only one mass scale. Consequently, we derive the critical exponents characterizing the anomalous dimensions of the field ϕ and composite operator ϕ^2 , respectively, within an ϵ_L -expansion at least at two-loop order using Feynman's path integral method in momentum space.

The anisotropic diagrams are calculated using the orthogonal approximation introduced in [10]. The isotropic diagrams are more flexible and can be calculated simultaneously using the orthogonal approximation as well as exactly. The exact computation was recently demonstrated in the massless theory [15] and shall be calculated here in the massive theory. The exponents computed diagrammatically in the present work agree with those obtained in the massless theory renormalized at nonvanishing external momenta as expected.

The organization of the paper is as follows. In Sec. II we present the normalization conditions for the anisotropic and isotropic Lifshitz critical behaviors in the massive case. The Callan-Symanzik equations and Wilson functions are defined in Sec. III. The critical exponents for the anisotropic cases are presented in Sec. IV. The computation of critical indices for the isotropic cases is the subject of Sec. V. The conclusions and perspectives of the present work are discussed in Sec. VI. The relevant Feynman integrals are calculated in the appendixes. Appendix A contains the information concerning the computations involved in the anisotropic behavior using an approximation to resolve the Feynman integrals. The isotropic integrals are computed using the same approximation in Appendix B. Finally, the exact isotropic integrals computed exactly are summarized in Appendix C.

II. NORMALIZATION CONDITIONS FOR THE MASSIVE THEORIES

Before setting the normalization conditions let us recall some basic facts about the Lifshitz critical behaviors. The bare Lagrangian density is that of a self-interacting scalar field given by [10] :

$$L = \frac{1}{2} |\nabla_m^2 \phi_0|^2 + \frac{1}{2} |\nabla_{(d-m)} \phi_0|^2 + \delta_0 \frac{1}{2} |\nabla_m \phi_0|^2 + \frac{1}{2} \mu_\tau^{2\tau} \phi_0^2 + \frac{1}{4!} \lambda_{0\tau} \phi_0^4. \quad (1)$$

The Lifshitz point, where $\mu_\tau = 0$ (for the temperature fixed at the Lifshitz value $T = T_L$) and $\delta_0 = 0$ is the multicritical point of interest in this framework. Consider the anisotropic criticalities. The first term in the above expression should be multiplied by a constant in order to make sense on dimensional grounds. Fortunately, the condition $\delta_0 = 0$, which will

be implemented in all the subsequent discussion, permits the dimensional redefinition of momentum scales along the competing m -dimensional axes with quartic momentum dependence. They have half the value of a genuine momentum scale and we can get rid of the constant in front of this unusual kinetic term. In the above Lagrangian we introduce two independent bare masses in order to perform the appropriate scale transformations consistent with the two independent correlation lengths characterizing these situations. They have different powers in order to emphasize that the momenta subspaces are inequivalent. Thus, the bare mass μ_2 , as well as its renormalized counterpart m_2 have half the value of a genuine mass scale. These distinctions do bring new insights for it is convenient that mass and momentum have the same canonical dimension in each subspace separately. In addition, two independent cutoffs are needed to provide independent flows in the parameter space as far as the masses are concerned. Together with two coupling constants characterizing each subspace, we have an apparent overcounting which allows the renormalization group transformations to be treated independently in the inequivalent subspaces. In the isotropic critical behavior there are some differences: the second term in the Lagrangian is absent and there is only one type of bare (and renormalized) mass(es).

It is interesting to consider the renormalized theories that can be constructed out of the Lagrangian (1). To do so, we shall follow the conventions adopted in [10]. The reader should also consult Ref.[8] for further notation details.

Let us turn our attention to the anisotropic cases. The two subspaces, namely one with a quadratic dependence on the momenta and the other with a quartic dependence on the momenta are orthogonal to each other and can be treated independently. In fact, we can define two sets of normalization conditions associated to either subspace as follows. The quadratic subspace is defined by $\mu_2, \lambda_2 = 0, \mu_1, \lambda_1 \neq 0$. The latter induce renormalized quantities m_1 and g_1 . Since no infrared divergence takes place in massive theories, we can renormalize the theory at zero external momenta. The one-particle irreducible(1PI)

renormalized vertex parts are defined through the following normalization conditions:

$$\Gamma_{R(1)}^{(2)}(0, m_1, g_1) = m_1^2, \quad (2a)$$

$$\frac{\partial \Gamma_{R(1)}^{(2)}(p, m_1, g_1)}{\partial p^2} \Big|_{p^2=0} = 1, \quad (2b)$$

$$\Gamma_{R(1)}^{(4)}(0, m_1, g_1) = g_1, \quad (2c)$$

$$\Gamma_{R(1)}^{(2,1)}(0, 0, m_1, g_1) = 1, \quad (2d)$$

$$\Gamma_{R(1)}^{(0,2)}(0, m_1, g_1) = 0. \quad (2e)$$

The subscript 1 makes explicit reference to the quadratic or noncompeting subspace. The most general vertex parts at this subspace have nonvanishing external momenta along the quadratic directions as well as zero external momenta components along the quartic directions.

Similarly, the definition of renormalized theories along the quartic or competing directions can be performed by setting first all the external momenta along the noncompeting directions equal to zero and taking $\mu_1, \lambda_1 = 0$ with $\mu_2, \lambda_2 \neq 0$. The resulting 1PI renormalized parts in the competing subspace can be rendered finite by using the following set of normalization conditions:

$$\Gamma_{R(2)}^{(2)}(0, m_2, g_2) = m_2^4, \quad (3a)$$

$$\frac{\partial \Gamma_{R(2)}^{(2)}(p, m_2, g_2)}{\partial p^4} \Big|_{p^4=0} = 1, \quad (3b)$$

$$\Gamma_{R(2)}^{(4)}(0, m_2, g_2) = g_2, \quad (3c)$$

$$\Gamma_{R(2)}^{(2,1)}(0, 0, m_2, g_2) = 1, \quad (3d)$$

$$\Gamma_{R(2)}^{(0,2)}(0, m_2, g_2) = 0. \quad (3e)$$

The subscript 2 refers to the competing subspace. Therefore, the competing and noncompeting subspaces are effectively separated out. In order to be completely similar to the massless case normalization conditions devised in Ref.[10], we can fix the mass scale of the two-point functions in either subspace by the choices $m_\tau^{2\tau} = 1$. Furthermore, the isotropic critical behavior can be defined using only the second set of normalization conditions from the anisotropic behaviors, with different values of renormalized mass and coupling constants, say, m_3, g_3 . Next, we apply these conditions to the discussion of the Callan-Symanzik equations for the anisotropic and isotropic m -axial critical behaviors.

III. THE CALLAN-SYMANZIK EQUATIONS

Let the label $\tau = 1, 2$ refer to the different external momenta scales involved in the general Lifshitz critical behavior, as discussed above for different normalization conditions in the anisotropic and isotropic cases. In terms of the bare quantities, the renormalized 1PI vertex parts are defined by

$$\Gamma_{R(\tau)}^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, g_\tau, m_\tau) = Z_{\phi(\tau)}^{\frac{N}{2}} Z_{\phi^2(\tau)}^L (\Gamma^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, \lambda_\tau, \mu_\tau, \Lambda_\tau) - \delta_{N,0} \delta_{L,2} \Gamma_{(\tau)}^{(0,2)}(Q_{(\tau)}, Q_{(\tau)}, \lambda_\tau, \mu_\tau, \Lambda_\tau)|_{Q_{(\tau)}^2=0}) \quad (4)$$

where $p_{i(\tau)}$ ($i = 1, \dots, N$) are the external momenta associated to the vertex functions $\Gamma_{R(\tau)}^{(N,L)}$ with N external legs, $Q_{i(\tau)}$ ($i = 1, \dots, L$) are the external momenta associated to the L insertions of ϕ^2 operators and Λ_τ are cutoffs characterizing each inequivalent subspace. In terms of the dimensionless couplings u_τ , the renormalized and bare coupling constants are given, respectively, by $g_\tau = u_\tau(m_\tau^{2\tau})^{\frac{\epsilon_L}{2}}$, and $\lambda_\tau = u_{0\tau}(m_\tau^{2\tau})^{\frac{\epsilon_L}{2}}$, where $\epsilon_L = 4 + \frac{m}{2} - d$. In addition, we can write all the renormalization functions in terms of u_τ .

By expanding the dimensionless bare coupling constants $u_{0\tau}$ and the renormalization functions $Z_{\phi(\tau)}$, $\bar{Z}_{\phi^2(\tau)} = Z_{\phi(\tau)} Z_{\phi^2(\tau)}$ in terms of the dimensionless renormalized couplings u_τ up to two-loop order as

$$u_{0\tau} = u_\tau(1 + a_{1\tau}u_\tau + a_{2\tau}u_\tau^2), \quad (5a)$$

$$Z_{\phi(\tau)} = 1 + b_{2\tau}u_\tau^2 + b_{3\tau}u_\tau^3, \quad (5b)$$

$$\bar{Z}_{\phi^2(\tau)} = 1 + c_{1\tau}u_\tau + c_{2\tau}u_\tau^2, \quad (5c)$$

we can unravel the renormalization structure of the Callan-Symanzik (CS) equations as they arise in the two situations. Let us now make a distinction in the two cases by considering separately the CS equations for anisotropic and isotropic spaces.

A. Anisotropic

In spite of having two independent mass scales, the asymptotic behaviors of the anisotropic cases at the critical dimension $d_c = 4 + \frac{m}{2}$ proceed along the same lines as the ordinary critical behavior at 4 dimensions described by a $\lambda\phi^4$ field theory. Therefore, we shall limit ourselves to the situation where $d = 4 + \frac{m}{2} - \epsilon_L$.

Before deriving explicitly the equations which relate theories renormalized at different mass scales, let us recall some useful definitions for certain quantities in terms of dimensionless coupling constants. The *beta*-functions and Wilson functions for the anisotropic behaviors are defined by:

$$\beta_\tau = (m_\tau \frac{\partial u_\tau}{\partial m_\tau}), \quad (6a)$$

$$\gamma_{\phi(\tau)}(u_\tau) = \beta_\tau \frac{\partial \ln Z_{\phi(\tau)}}{\partial u_\tau} \quad (6b)$$

$$\bar{\gamma}_{\phi^2(\tau)}(u_\tau) = -\beta_\tau \frac{\partial \ln \bar{Z}_{\phi^2(\tau)}}{\partial u_\tau} \quad (6c)$$

are calculated at fixed bare coupling λ_τ . Since the theory is massive, scale invariance will be achieved in the limit where the cutoffs Λ_τ go to infinity. The β_τ -functions can be cast in a more useful form in terms of dimensionless quantities, namely,

$$\beta_\tau = -\tau \epsilon_L (\frac{\partial \ln u_{0\tau}}{\partial u_\tau})^{-1}. \quad (7)$$

By differentiating Eq.(4) with respect to $\ln m_\tau$ we find

$$\begin{aligned} & (m_\tau \frac{\partial}{\partial m_\tau} + \beta_\tau \frac{\partial}{\partial g_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(u_\tau) + L \gamma_{\phi^2(\tau)}(u_\tau)) \Gamma_{R(\tau)}^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, g_\tau, m_\tau) \\ & - \delta_{N,0} \delta_{L,2} (\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}} B_\tau = m_\tau^{2\tau} (2\tau - \gamma_{\phi(\tau)}) \Gamma_{R(\tau)}^{(N,L+1)}(p_{i(\tau)}, Q_{i(\tau)}, 0, g_\tau, m_\tau) \end{aligned} \quad (8)$$

where B_τ is a constant used to renormalize $\Gamma_{R(\tau)}^{(0,2)}$ and $\gamma_{\phi^2(\tau)}(u_\tau) = -\beta_\tau \frac{\partial \ln Z_{\phi^2(\tau)}}{\partial u_\tau}$. The last equations are the Callan-Symanzik equations for anisotropic vertex parts with arbitrary composite operators. The right hand side can be asymptotically neglected order by order in perturbation theory. Consequently, the Callan-Symanzik (CS) equations have asymptotically the same form as the renormalization group (RG) equations previously derived in Ref.[10].

For the sake of simplicity, consider the vertex parts without composite operators with $L = 0$. A dimensional redefinition of the momentum components along the competing subspace has been performed (see the discussion following Eq.(1) in the last section). Since the resulting “effective” space dimension for the anisotropic cases is $d_{eff} = d - \frac{m}{2}$, where d is the space dimension of the system, dimensional analysis yields

$$\Gamma_{R(\tau)}^{(N)}(\rho_\tau k_{i(\tau)}, u_\tau, m_\tau) = \rho_\tau^{\tau(N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2})} \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, u_\tau(\rho_\tau), \frac{m_\tau}{\rho_\tau}).$$

Now consider the asymptotic part of the corresponding vertex function satisfying the homogeneous CS equation. The solution is given by:

$$\Gamma_{as\ R(\tau)}^{(N)}(k_{i(\tau)}, u_\tau, \frac{m_\tau}{\rho_\tau}) = \exp\left[-\frac{N}{2} \int_{u_\tau}^{u_\tau(\rho_\tau)} \gamma_{\phi(\tau)}(u'_\tau(\rho_\tau)) \frac{du'_\tau}{\beta_\tau(u'_\tau)}\right] \Gamma_{as\ R(\tau)}^{(N)}(k_{i(\tau)}, u_\tau(\rho_\tau), m_\tau), \quad (9)$$

where

$$\rho_\tau = \int_{u_\tau}^{u_\tau(\rho_\tau)} \frac{du'_\tau}{\beta_\tau(u'_\tau)}. \quad (10)$$

From now on we shall drop the subscript for asymptotic in the vertex parts satisfying the homogeneous Callan-Symanzik equations. The values of the coupling constants $u_{\tau\infty}$ yielding the eigenvalue conditions $\beta_\tau(u_{\tau\infty}) = 0$ do exist, and precisely at this point the solutions to the CS equations under a scale in the external momenta obey the equations

$$\Gamma_{R(\tau)}^{(N)}(\rho_\tau k_{i(\tau)}, u_{\tau\infty}, m_\tau) = \rho_\tau^{\tau(N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2})-\frac{N\gamma_{\phi(\tau)}(u_{\tau\infty})}{2}} \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, u_{\tau\infty}, m_\tau). \quad (11)$$

The dimension of the field is defined by

$$\Gamma_{(\tau)}^{(N)}(\rho_\tau k_{i(\tau)}) = \rho_\tau^{\tau[(d-\frac{m}{2})-Nd_{\phi(\tau)}]} \Gamma_{(\tau)}^{(N)}(k_i). \quad (12)$$

This in turn implies that the definition of the anomalous dimension of the field through $d_{\phi(\tau)} = \frac{d-\frac{m}{2}}{2} - 1 + \frac{\eta_\tau}{2\tau}$ corresponds to $\eta_\tau = \gamma_{\phi(\tau)}(u_{\tau\infty})$.

If we consider the vertex parts including composite operators, we can identify the anomalous dimension of the composite operator ϕ^2 as follows. For $u_\tau = u_{\tau\infty}$ the asymptotic behavior of the vertex parts are given by $((N, L) \neq (0, 2))$

$$\Gamma_{R(\tau)}^{(N,L)}(\rho_\tau k_{i(\tau)}, \rho_\tau p_{i(\tau)}, u_{\tau\infty}, m_\tau) = \rho_\tau^{\tau(N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2}-2L)-\frac{N\gamma_{\phi(\tau)}(u_{\tau\infty})}{2}+L\gamma_{\phi^2(\tau)}(u_{\tau\infty})} \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, p_{i(\tau)}, u_{\tau\infty}, m_\tau). \quad (13)$$

Writing the coefficient in the right hand side as $\rho_\tau^{\tau[(d-\frac{m}{2})-Nd_{\phi(\tau)}]+Ld_{\phi^2(\tau)}}$, we conclude that $d_{\phi^2(\tau)} = -2\tau + \gamma_{\phi^2(\tau)}(u_{\tau\infty})$. The correlation length exponents can now be obtained through the relation $\nu_\tau^{-1} = -d_{\phi^2(\tau)} = 2\tau - \gamma_{\phi^2(\tau)}(u_{\tau\infty})$. We now have the resources to calculate these two exponents diagrammatically.

B. Isotropic

Let us briefly discuss the isotropic case $d = m$. Owing to the existence of only one type of scaling transformation, all we have to do is to concentrate on the competing sector ($\tau = 2$) of the anisotropic treatment performed above. That means that we start with the mass scale m_3 , dimensionless coupling constant u_3 and subscript 3 in all vertex functions in order to avoid confusion with the competing subspace of the anisotropic cases. There are 3 main differences. First, the critical dimension is 8. Therefore, the expansion parameter is $\epsilon_L = 8 - m$. Second, the effective space dimension is $\frac{m}{2}$. Finally, the beta function does not have the global factor of two as in the competition directions of the anisotropic case. Instead, in terms of dimensionless parameters one has $\beta_3 = -\epsilon_L(\frac{\partial \ln u_{03}}{\partial u_3})^{-1}$. Following the same trend as before, we conclude that the anomalous dimension of the field in this case is $\eta_{L4} \equiv \eta_3 = \gamma_{\phi(3)}(u_{3\infty})$, where $u_{3\infty}$ is the value of the coupling constant which yields $\beta_3(u_{3\infty}) = 0$. In addition, the anomalous dimension of the composite operator is related to the correlation length exponent through the relation $\nu_3^{-1} = -d_{\phi^2(3)} = 4 - \gamma_{\phi^2(3)}(u_{3\infty})$. Let us turn our attention to the calculation of these critical exponents by means of diagrammatic expansions.

IV. ANISOTROPIC CRITICAL EXPONENTS

We shall perform the computation of the critical exponents using the normalization conditions (previously introduced in Sec.II) for the anisotropic cases.

We start by using eqs.(2),(3) and (5) in order to determine the normalization functions $Z_{\phi(\tau)}, \bar{Z}_{\phi^2(\tau)}$ in powers of u_τ along with the Feynman integrals at the loop order required. We employ the orthogonal approximation in the calculation of Feynman integrals. The resulting expressions for the normalization functions can be written in terms of the four diagrams I_2, I'_3, I_4, I'_5 of one-, two- and three-loop order, respectively. They are presented in Appendix A. We recall that each loop integral produces a geometric angular factor $\frac{1}{4}S_m S_{d-m}\Gamma(2 - \frac{m}{4})\Gamma(\frac{m}{4})$, which shall be absorbed in a redefinition of the coupling constants. Performing those redefinitions, we obtain the renormalization functions in terms of the above mentioned

diagrams, namely:

$$u_{0\tau} = u_\tau \left[1 + \frac{(N+8)}{6} I_2 u_\tau + \left(\frac{[(N+8)I_2]^2}{18} - \left(\frac{(N^2+6N+20)I_2^2}{36} + \frac{(5N+22)I_4}{9} \right) - \frac{(N+2)I_3'}{9} \right) u_\tau^2 \right], \quad (14a)$$

$$Z_{\phi(\tau)} = 1 + \frac{(N+2)I_3'}{18} u_\tau + \frac{(N+2)(N+8)(I_2 I_3' - \frac{I_5'}{2})}{54} u_\tau^2, \quad (14b)$$

$$\bar{Z}_{\phi^2(\tau)} = 1 + \frac{(N+2)I_2}{6} u_\tau + \left[\frac{(N^2+7N+10)I_2}{18} - \frac{(N+2)}{6} \left(\frac{(N+2)I_2^2}{6} + I_4 \right) \right] u_\tau^2. \quad (14c)$$

As usual, the coefficients of the various terms in powers of u_τ have poles in ϵ_L . These poles cancel in the calculation of the β_τ and the critical exponents. In fact using Eqs.(5), the β_τ and Wilson functions in either subspace are given by:

$$\beta_\tau = -\tau \epsilon_L u_\tau [1 - a_{1\tau} u_\tau + 2(a_{1\tau}^2 - a_{2\tau}) u_\tau^2], \quad (15a)$$

$$\gamma_{\phi(\tau)} = -\tau \epsilon_L u_\tau [2b_{2\tau} u_\tau + (3b_{3\tau} - 2b_{2\tau} a_{1\tau}) u_\tau^2], \quad (15b)$$

$$\bar{\gamma}_{\phi^2(\tau)} = \tau \epsilon_L u_\tau [c_{1\tau} + (2c_{2\tau} - c_{1\tau}^2 - a_{1\tau} c_{1\tau}) u_\tau]. \quad (15c)$$

Substitution of the explicit values of the coefficients given in (13) along with the results for the diagrams computed in Appendix A yields the following expression for β_τ

$$\begin{aligned} \beta_\tau = & -\tau u_\tau \left[\epsilon_L - \frac{(N+8)}{6} (1 + ([i_2]_m - 1)) u_\tau \right. \\ & \left. - \frac{(3N+14)}{12} u_\tau^2 \right] + O(u_\tau^4). \end{aligned} \quad (16)$$

From this expression, a zero of order ϵ_L of each β_τ characterizing the independent non-competing ($\tau = 1$) and competing ($\tau = 2$) subspaces correspond to the same value of the coupling constant ($u_{1\infty} = u_{2\infty} \equiv u_\infty$), i.e.,

$$u_\infty = \frac{6}{8+N} \epsilon_L \left\{ 1 + \epsilon_L \left[-([i_2]_m - 1) + \frac{(9N+42)}{(8+N)^2} \right] \right\}. \quad (17)$$

Replacing this value back in the functions $\gamma_{\phi(\tau)}$ and $\bar{\gamma}_{\phi^2(\tau)}$ together with the coefficients given in (14) and in Appendix A, we find

$$\gamma_{\phi(\tau)}(u_\infty) = \frac{\tau}{2} \epsilon_L^2 \frac{N+2}{(N+8)^2} \left[1 + \epsilon_L \left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4} \right) \right]. \quad (18)$$

Notice that these expression are exactly the same of those for the critical exponents $\eta_{L2} \equiv \eta_1$ and $\eta_{L4} \equiv \eta_2$ obtained previously in [10] upon the identification $\gamma_{\phi(1)}(u_\infty) = \eta_{L2}$ and $\gamma_{\phi(2)}(u_\infty) = \eta_{L4}$.

The exponents ν_τ can be found analogously from the expression of the anomalous dimension of the composite operator ϕ^2 . Recalling that $\gamma_{\phi^2(\tau)} = \bar{\gamma}_{\phi^2(\tau)} + \gamma_{\phi(\tau)}$, it follows that $\nu_\tau^{-1} = -d_{\phi^2(\tau)} = 2\tau - \gamma_{\phi^2(\tau)}(u_{\tau\infty}) - \gamma_{\phi(\tau)}(u_\infty)$. Using the results in Appendix A once again, we obtain the following intermediate step:

$$\bar{\gamma}_{\phi^2(\tau)}(u_\tau) = \tau \frac{(N+2)}{6} u_\tau [1 + \epsilon_L([i_2]_m - 1) - \frac{1}{2} u_\tau]. \quad (19)$$

Then, inserting the value u_∞ into the last expression as well as $\gamma_{\phi(\tau)}(u_\infty)$ from Eq.(18), we get

$$\nu_\tau = \frac{1}{2\tau} + \frac{(N+2)}{4\tau(N+8)} \epsilon_L + \frac{1}{8\tau} \frac{(N+2)(N^2 + 23N + 60)}{(N+8)^3} \epsilon_L^2. \quad (20)$$

These exponents are in perfect agreement with those calculated previously using a massless theory [10] when we identify $\nu_1 = \nu_{L2}$ and $\nu_2 \equiv \nu_{L4}$, respectively. The other exponents can be obtained from these (computed diagrammatically) via scaling relations.

V. ISOTROPIC CRITICAL EXPONENTS

We are going to use the main results derived in the previous section for the normalization functions in the presence of a single mass parameter and only one type of length/momentum scale corresponding to the isotropic critical behavior. In this case, each loop integral produces an angular factor of S_m , the area of an m -dimensional unit sphere, which shall be absorbed in a redefinition of the coupling constant. First, we describe the exponents using the orthogonal approximation for Feynman diagrams. Then, we shall present the results without the need for approximations in calculating the graphs, which is a simple feature of isotropic points.

A. Exponents in the orthogonal approximation

The basic definitions of the dimensionless renormalized coupling constant and normalization functions can be retrieved from Eqs.(14) using the subscript $\tau = 3$ in order not to confuse with the competing sector of the anisotropic cases. Moreover, the Feynman integrals appropriate for the isotropic case have to be used. The results of these diagrams

using the orthogonal approximation are given in Appendix B, which are going to be used in this subsection. Furthermore, the functions obtained from the normalization constants and dimensionless coupling constant through Eqs.(6) in the isotropic cases now read

$$\beta_3 = -\epsilon_L u_3 [1 - a_{13} u_3 + 2(a_{13}^2 - a_{23}) u_3^2], \quad (21a)$$

$$\gamma_{\phi(3)} = -\epsilon_L u_3 [2b_{23} u_3 + (3b_{33} - 2b_{23} a_{13}) u_3^2], \quad (21b)$$

$$\bar{\gamma}_{\phi^2(3)} = \epsilon_L u_3 [c_{13} + (2c_{23} - c_{13}^2 - a_{13} c_{13}) u_3]. \quad (21c)$$

Recall that in the above expressions $\epsilon_L = 8 - m$. Using the integrals presented in Appendix B, we find

$$\begin{aligned} \beta_3 = & -u_3 \left[\epsilon_L - \frac{(N+8)}{6} \left(1 - \frac{1}{4} \epsilon_L \right) u_3 \right. \\ & \left. + \frac{(3N+14)}{24} u_3^2 \right] + O(u_3^4). \end{aligned} \quad (22)$$

From the eigenvalue condition $\beta_3(u_{3\infty}) = 0$ we obtain the following result for the zero of order ϵ_L of the dimensionless coupling constant

$$u_{3\infty} = \frac{6}{8+N} \epsilon_L \left\{ 1 + \frac{\epsilon_L}{2} \left[\frac{1}{2} + \frac{(9N+42)}{(8+N)^2} \right] \right\}. \quad (23)$$

Using this value in the expression for $\gamma_{\phi(3)}$ in conjunction with Eqs.(14) and the results in Appendix B, one can show that

$$\gamma_{\phi(3)}(u_{3\infty}) = \epsilon_L^2 \frac{N+2}{4(N+8)^2} \left[1 + \epsilon_L \left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{8} \right) \right]. \quad (24)$$

Once more, this is precisely the value of the exponent η_{L4} previously obtained using the renormalization group equation in the massless theory [10]. Moreover, using again the results in Appendix B in the definition of $\bar{\gamma}_{\phi^2(3)}$, we obtain

$$\bar{\gamma}_{\phi^2(3)}(u_3) = \frac{(N+2)}{6} u_3 \left[1 - \frac{1}{4} \epsilon_L - \frac{1}{4} u_3 \right]. \quad (25)$$

When Eq.(23) is replaced into this expression it yields

$$\bar{\gamma}_{\phi^2(3)}(u_{3\infty}) = \frac{(N+2)}{(N+8)} \epsilon_L \left[1 + \frac{(3N+9)}{(N+8)^2} \epsilon_L \right], \quad (26)$$

and recalling that $\nu_3 \equiv \nu_{L4} = (4 - \bar{\gamma}_{\phi^2(3)}(u_{3\infty}) - \gamma_{\phi(3)}(u_{3\infty}))^{-1}$, we get to

$$\nu_{L4} = \frac{1}{4} + \frac{(N+2)}{16(N+8)} \epsilon_L + \frac{1}{256} \frac{(N+2)(N^2+23N+60)}{(N+8)^3} \epsilon_L^2. \quad (27)$$

The expression for $\bar{\gamma}_{\phi^2(3)}(u_{3\infty})$ in Eq.(26) is the same as the one associated to a scalar theory in the massless limit, computed at the fixed point using the renormalization group equation. Besides, Eq.(27) corresponds to Eq.(208) for this exponent in the orthogonal approximation using the method of Ref.[10].

B. Exponents in the exact computation

The Feynman integrals of the isotropic case can be manipulated without referring to any approximation in the massive case, as described in Appendix C. Our goal here is to rederive the exponents using the machinery developed in the last subsection replacing, however, the expression for the diagrams by their exact counterparts presented in Appendix C.

We start by writing down the β_3 function explicitly in terms of the diagrams calculated in Appendix C, which in that case reduces to

$$\begin{aligned} \beta_3 = & -u_3[\epsilon_L - \frac{(N+8)}{6}(1 - \frac{1}{4}\epsilon_L)u_3 \\ & - \frac{(41N+202)}{1080}u_3^2] + O(u_3^4). \end{aligned} \quad (28)$$

Therefore, this expression has a zero at the following value of the dimensionless coupling constant

$$u_{3\infty} = \frac{6}{8+N} \epsilon_L \left\{ 1 + \frac{\epsilon_L}{2} \left[\frac{1}{2} - \frac{(41N+202)}{15(8+N)^2} \right] \right\}. \quad (29)$$

Following the same steps as before it is not difficult to show that this value of the coupling constant yields

$$\gamma_{\phi(3)}(u_{3\infty}) = -\epsilon_L^2 \frac{3(N+2)}{20(N+8)^2} + \frac{(N+2)}{10(N+8)^2} \left[\frac{(41N+202)}{10(N+8)^2} + \frac{23}{80} \right]. \quad (30)$$

This expression is equal to the value of the exponent η_{L4} calculated exactly in conformity with that of $O(\epsilon_L^2)$ firstly found by Hornreich, Luban and Shtrikman [1] and recently confirmed and extended to $O(\epsilon_L^3)$ in Ref.[15] for the massless case, as expected.

Finally, let us determine ν_{L4} . First, we have to make use of the results in Appendix C to write down the following expression for arbitrary dimensionless coupling constant

$$\bar{\gamma}_{\phi^2(3)}(u_3) = \frac{(N+2)}{6} u_3 \left[1 - \frac{1}{4}\epsilon_L + \frac{1}{12}u_3 \right]. \quad (31)$$

Second, we replace the especial value of the coupling constant $u_{3\infty}$ in this equation in order to show that

$$\bar{\gamma}_{\phi^2(3)}(u_{3\infty}) = \frac{(N+2)}{(N+8)}\epsilon_L \left[1 - \frac{(13N+41)}{15(N+8)^2}\epsilon_L\right], \quad (32)$$

which allows us to figure out the value of ν_{L4} from $\gamma_{\phi(3)}(u_{3\infty})$ and $\bar{\gamma}_{\phi^2(3)}(u_{3\infty})$, namely

$$\nu_{L4} = \frac{1}{4} + \frac{(N+2)}{16(N+8)}\epsilon_L + \frac{(N+2)(15N^2+89N+4)}{960(N+8)^3}\epsilon_L^2. \quad (33)$$

This is precisely the exponent ν_{L4} in the exact calculation recently found at $O(\epsilon_L^2)$ [15]. As before, the other exponents can be discovered through the use of scaling relations.

VI. DISCUSSION OF THE RESULTS AND CONCLUSIONS

It is interesting to find two different methods producing the same values of the critical exponents calculated diagrammatically. Those previously computed using a massless scalar field theory subtracted at nonzero external momenta using the renormalization group equations in Refs.[10] are rederived herein using massive theories (with two or one independent mass scales depending whether we treat the anisotropic or isotropic cases) in the Callan-Symanzik method.

Most of the remarks done originally by Brezin, Le Guillou and Zinn-Justin in Ref.[14] for conventional critical phenomena are still valid here, for isotropic as well as anisotropic cases. First, the eigenvalue condition $\beta_\tau(u_{\tau\infty}) = 0$ corresponds to a scale invariant theory, as shown here, but it is not attractive: the theory is only scale invariant precisely at this point, since the derivative $\beta'_\tau(u_{\tau\infty})$ calculated at this point is positive. Second, our calculations are valid in the region of momenta much larger than the mass in any particular subspace. The difference dwells in the anisotropic cases: two inequivalent subspaces require two independent mass scales along with two coupling constants which consistently have the same eigenvalue leading to scale invariance separately in each subspace. Universality is recovered when the renormalized dimensionless coupling constants are fixed at the solution of the eigenvalue $u_{\tau\infty}$.

One step further within the formalism introduced in the present work is to extend it in order to treat arbitrary competing systems, when arbitrary types of competing directions characterized by higher order derivative terms of arbitrary powers are involved[15], generalizing the present concept. The Feynman integrals get much more difficult to solve,

especially when one tries to perform the exact calculation for the isotropic cases. That is the reason we have calculated here the integrals using the orthogonal approximation and the exact computation for those integrals. In spite of yielding different values for the critical exponents, the difference among them in each calculation is negligible as already pointed out in [15]. We shall leave the subject of generic competing systems in the Callan-Symanzik method for future work [16].

The anisotropic behavior with two independent mass scales and the analysis developed herein might be useful in addressing further anisotropic problems in quantum field theory, possibly with broken Lorentz/Poincaré invariance. First, it is useful to consider the higher order terms in momentum space appearing isotropically in the ultraviolet regime. This possibility arises, for instance, in Lorentz-violating dispersion relations for cosmic ray processes due to the nontrivial character of the short-distance structure of spacetime [17]. This effect can be implemented in the propagation of massive particles as well, even anisotropically. One mathematical possibility to describe such phenomena is that the metric tensor depends upon the momentum variables. This has been done recently to investigate the simplest example in anisotropic Planck-scale modified dispersion relations for a relativistic particle of mass m propagating in 2 dimensions taking into account phenomenological quantum gravitational effects (i.e, the modified dispersion relations) in the framework of Finsler geometry. The coefficient of the cubic term in the momentum p_1 in the mass-shell condition is directly related to the Planck mass M [18]. In principle, these modified dispersion relations can be postulated for massive fields as well. Then the procedure should be the opposite to that described in the present paper to obtain the Lagrangian: keep $\delta_0 \neq 0$, restore the dimensionful character of the higher order derivative terms and introduce the equations of motion with any power of momenta (mass-shell condition) isotropically or anisotropically, as a constraint using an auxiliary field (Lagrange multiplier). The problem is that the equation of motion obtained by varying the Lagrange multiplier is not easy to solve (even in the particle case) and the renormalization group structure in this framework is not obvious.

Another perspective is to follow the trend proposed here of maintaining $\delta_0 = 0$ which allows the decoupling of the inequivalent subspaces and makes it easier the ultraviolet renormalization of quantum fields. Siegel's recipe to introduce masses by dimensional reduction in Poincaré invariant field theories [19] would be modified in the present case with at least two inequivalent subspaces. This would require two extra dimensions to characterize completely

the two mass scales involved in the problem, one for each momentum subspace. This is so because one can mathematically formulate a gradient operator for “Lifshitz spaces” which can be defined as $\partial_i \equiv (\partial_1, \dots, \partial_{(d-m)}, \partial_{new(d-m+1)} \equiv \partial_{old(d-m+1)}^2, \dots, \partial_{new(d)} \equiv \partial_{old(d)}^2)$ through the dimensional redefinition proposed in [10], with the same Euclidean metric, keeping in mind that the mass m_1 characterizes the noncompeting (quadratic) subspace, while m_2 is the typical scale of the competing (quartic) subspace, with the latter having half of the canonical dimension of the former. This new operator would reflect directly the “folding” of the dimensions along the competing subspace as a dynamical effect provoked by the combination of attractive and repulsive components of the background on which quantum fields propagate. This new flat space limit does not correspond to Minkowski space. Although this is trivial for scalar fields, it is definitely not the case for defining spinors and vector fields in these Lifshitz spaces with anisotropic higher derivative terms in the kinetic part.

In conclusion, the new method proposed in the present work to study m -axial Lifshitz points for the massive scalar field theory renormalized at zero external momenta opens the possibility to study field theories with inequivalent subspaces characterized by two (or more) mass scales. Universality is confirmed, since the critical exponents calculated in the massive theory using the Callan-Symanzik formalism agree with those obtained in the massless scalar field theory renormalized at nonvanishing external momenta.

VII. ACKNOWLEDGMENTS

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APPENDIX A: FEYNMAN GRAPHS FOR ANISOTROPIC BEHAVIORS

We compute the appropriate diagrams corresponding to one-, two- and three-loop integrals using dimensional regularization and Feynman parameters to fold several denominators together. Only the one-loop integral for the four-point function can be solved exactly for anisotropic criticalities. The higher loop diagrams can be calculated, however, using the orthogonal approximation, which is the most general approximation consistent with the physical property of homogeneity. The basic statement of this approximation is that the

loop momenta in a given subdiagram is orthogonal to all loop momenta appearing in other subdiagrams. Due to our normalization conditions defined in the text, we shall work with zero external momenta. Explicitly, the integrals we have to resolve are given by:

$$I_2 = \int \frac{d^{d-m}q d^m k}{[(k)^2 + (q)^2 + 1]^2} , \quad (A1)$$

$I'_3 = \frac{\partial I_3(P,K)}{\partial P^2}|_{P=K=0} (= \frac{\partial I_3(P,K)}{\partial K^4}|_{P=K=0})$ where $I_3(P, K)$ is the integral

$$I_3 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{(q_1^2 + (k_1^2)^2 + 1)(q_2^2 + (k_2^2)^2 + 1)[(q_1 + q_2 + P)^2 + ((k_1 + k_2 + K)^2)^2 + 1]} , \quad (A2)$$

$$I_4 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{(q_1^2 + (k_1^2)^2 + 1)^2 (q_2^2 + (k_2^2)^2 + 1)} \times \frac{1}{[(q_1 + q_2)^2 + ((k_1 + k_2)^2)^2 + 1]} , \quad (A3)$$

and $I'_5 = \frac{\partial I_5(P,K)}{\partial P^2}|_{P=K=0} (= \frac{\partial I_5(P,K)}{\partial K^4}|_{P=K=0})$ where $I_5(P, K)$ is the three-loop diagram

$$I_5 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^{d-m}q_3 d^m k_1 d^m k_2 d^m k_3}{((q_1 + P)^2 + ((k_1 + K)^2)^2 + 1)(q_2^2 + (k_2^2)^2 + 1)(q_3^2 + (k_3^2)^2 + 1)} \times \frac{1}{[(q_1 + q_2)^2 + ((k_1 + k_2)^2)^2 + 1][(q_1 + q_3)^2 + ((k_1 + k_3)^2)^2 + 1]} . \quad (A4)$$

Proceeding as indicated above, their ϵ_L -expansion have the following expressions

$$I_2 = \frac{1}{\epsilon_L} [1 + ([i_2]_m - 1)\epsilon_L] + O(\epsilon_L), \quad (A5)$$

$$I'_3 = \frac{-1}{8\epsilon_L} [1 + (2[i_2]_m - \frac{5}{4})\epsilon_L] - \frac{1}{8}I + O(\epsilon_L), \quad (A6)$$

$$I_4 = \frac{1}{2\epsilon_L^2} \left(1 + (2[i_2]_m - \frac{3}{2})\epsilon_L + O(\epsilon_L^2) \right), \quad (A7)$$

$$I'_5 = \frac{-1}{6\epsilon_L^2} [1 + (3[i_2]_m - \frac{7}{4})\epsilon_L + O(\epsilon_L^2)] - \frac{1}{4\epsilon_L}I, \quad (A8)$$

where $[i_2]_m = 1 + \frac{1}{2}(\psi(1) - \psi(2 - \frac{m}{4}))$ and

$$I = \int_0^1 dx \left(\frac{1}{1 - x(1 - x)} + \frac{\ln[x(1 - x)]}{[1 - x(1 - x)]^2} \right), \quad (A9)$$

is the same integral appearing in the original work by BLZ. This integral disappears in the calculation of the critical exponents. Its appearance here is a general feature of the orthogonal approximation, since it will be present in the final result of the calculation of the diagrams in the isotropic cases as well. It does not appear in the massless theory and is only an artifact of our choice of the normalization point for the external momenta.

APPENDIX B: ISOTROPIC DIAGRAMS IN THE GENERALIZED ORTHOGONAL APPROXIMATION

We shall now turn our attention to the isotropic integrals in an expansion in $\epsilon_L = 8 - d$. The simplifying feature of this case is the existence of only one type of correlation length, which implies only one mass as usual in field theory calculations. The issue here is a quartic power in the propagator. The expressions for these graphs are:

$$I_2 = \int \frac{d^m k}{[(k^2)^2 + 1]^2} \quad , \quad (\text{B1})$$

$I'_3 = \frac{\partial I_3(P, K)}{\partial K^4} \big|_{K=0}$ where $I_3(K)$ is the integral

$$I_3 = \int \frac{d^m k_1 d^m k_2}{((k_1^2)^2 + 1) ((k_2^2)^2 + 1) [(k_1 + k_2 + K)^2 + 1]} \quad , \quad (\text{B2})$$

$$I_4 = \int \frac{d^m k_1 d^m k_2}{((k_1^2)^2 + 1)^2 ((k_2^2)^2 + 1)} \times \frac{1}{[(k_1 + k_2)^2 + 1]} \quad , \quad (\text{B3})$$

and $I'_5 = \frac{\partial I_5(K)}{\partial K^4} \big|_{K=0}$ where $I_5(K)$ is the three-loop diagram

$$I_5 = \int \frac{d^m k_1 d^m k_2 d^m k_3}{(((k_1 + K)^2)^2 + 1) ((k_2^2)^2 + 1) ((k_3^2)^2 + 1)} \times \frac{1}{[(k_1 + k_2)^2 + 1][(k_1 + k_3)^2 + 1]} \quad . \quad (\text{B4})$$

Using the same steps as indicated in the orthogonal approximation for the anisotropic behaviors we find for their ϵ_L -expansion the following results

$$I_2 = \frac{1}{\epsilon_L} [1 - \frac{1}{4}\epsilon_L] + O(\epsilon_L), \quad (\text{B5})$$

$$I'_3 = \frac{-1}{16\epsilon_L} [1 - \frac{1}{8}\epsilon_L] - \frac{1}{32}I + O(\epsilon_L), \quad (\text{B6})$$

$$I_4 = \frac{1}{2\epsilon_L^2} \left(1 - \frac{1}{4}\epsilon_L + O(\epsilon_L^2) \right), \quad (\text{B7})$$

$$I'_5 = \frac{-1}{12\epsilon_L^2} [1 - \frac{1}{8}\epsilon_L + O(\epsilon_L^2)] - \frac{1}{16\epsilon_L}I, \quad (\text{B8})$$

where I is the same integral which appears in Appendix A. As before this integral do not contribute for the critical exponents.

APPENDIX C: ISOTROPIC INTEGRALS IN THE EXACT CALCULATION

We shall consider now the integrals without performing approximations. The expressions are the same of those given in the previous appendix. Before quoting the results some commentaries are in order. First, the simplicity of the isotropic cases allow us to perform the integrals exactly as advertised. A useful trick is to write the propagators in the form $\frac{1}{[(k^2)^2+1]} = \frac{1}{(k^2+i)(k^2-i)}$ and then use a Feynman parameter. One can proceed as before since now the propagators are quadratic in the loop momenta, but the resulting parametric integrals turns out to be more complicated than those in the orthogonal approximation. Moreover, some care must be exercised in order to manipulate the remaining factors of i . Setting $\epsilon_L = 0$ in the powers of i when performing the parametric integrals effectively suppress it. We can then figure out the final expressions for the exact calculation, namely

$$I_2 = \frac{1}{\epsilon_L} \left[1 - \frac{1}{4} \epsilon_L \right] + O(\epsilon_L), \quad (C1)$$

$$I_3' = \frac{3}{80\epsilon_L} \left[1 - \frac{77}{120} \epsilon_L \right] - \frac{9}{2} H + O(\epsilon_L), \quad (C2)$$

$$I_4 = \frac{1}{2\epsilon_L^2} \left(1 - \frac{7}{12} \epsilon_L + O(\epsilon_L^2) \right), \quad (C3)$$

$$I_5' = \frac{1}{20\epsilon_L^2} \left[1 - \frac{13}{15} \epsilon_L + O(\epsilon_L^2) \right] - \frac{9}{\epsilon_L} H, \quad (C4)$$

and now the multiparametric integral is specific to m -axial isotropic Lifshitz points and turns out to be

$$\begin{aligned} H = & \int_0^1 dz z(1-z) \int_0^1 \int_0^1 \int_0^1 dx dy dt \int_0^1 du u(1-u)^3 \\ & \times \ln \left(u \left[\frac{2x(1-z) + 2yz - 1}{z(1-z)} - 2t + 1 \right] + 2t - 1 \right). \end{aligned} \quad (C5)$$

The integral H (as well as the integral I appearing in the orthogonal approximation) does not contribute to the critical exponents since it disappears in the renormalization algorithm.

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- [1] R. M. Hornreich, M. Luban and S. Shtrikman, *Phys. Rev. Lett.* **35**, 1678(1975).
 - [2] R. M. Hornreich, *Journ. Magn. Magn. Mat.* **15-18**, 387(1980).
 - [3] W. Selke, *Phys. Rep.* **170**, 213(1988); in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, London, 1998), Vol.15.
 - [4] N. Arkani-Hamed, H. Cheng, M. A. Luty and S. Mukohyama, *J. High Energy Phys.* **0405** 074 (2004); N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, *J. Cosmol. Astropart. Phys.* **0404** 001 (2004).
 - [5] N. Arkani-Hamed, H. Cheng, M. A. Luty and J. Thaler, *J. High Energy Phys.* **0507** 029 (2005); N. Arkani-Hamed, H. Cheng, M. A. Luty, S. Mukohyama and T. Wiseman, *hep-ph/0507120*; H. Cheng, M. A. Luty, S. Mukohyama and J. Thaler, *hep-th/0603010*; see also first paper from Ref.[4].
 - [6] K. G. Wilson and M. E. Fisher, *Phys. Rev. Lett.* **28**, 240 (1972); K. G. Wilson, *Phys. Rev. Lett.* **28**, 548 (1972); K. G. Wilson and J. Kogut, *Phys. Rep.* **12**, 75 (1974).
 - [7] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. A. Green (Academic Press, London, 1976), Vol.6.
 - [8] See also D. J. Amit, in *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).
 - [9] The mass in quantum field theory corresponds to the inverse correlation length, which diverges in the massless limit. Consequently, the two inequivalent $(d - m)$ - and m -dimensional subspaces define independent anisotropic renormalization scales characterized by two independent correlation lengths ξ_{L2} and ξ_{L4} , whereas if $m = d$ there is only one isotropic space associated with ξ_{L4} . See Ref.[10].
 - [10] M. M. Leite, *Phys. Rev. B* **67**, 104415 (2003).
 - [11] For an alternative field-theoretic treatment for m -axial Lifshitz points see M. Shpot, and H. W. Diehl, *Nucl. Phys. B* **612**(3), 340(2001); H. W. Diehl, and M. Shpot, *J. Phys. A: Math. Gen.* **35**, 6249(2002).
 - [12] C. G. Callan, Jr., *Phys. Rev. D* **2**, 1541 (1970).
 - [13] K. Symanzik, *Comm.. Math. Phys.* **18**, 227 (1970).
 - [14] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, *Phys. Rev. D* **8**, 434 (1973).

- [15] M. M. Leite, *Phys. Rev. B* **72**, 224432 (2005).
- [16] P. R. S. Carvalho and M. M. Leite, work in progress.
- [17] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, *Nature* **393**, 763 (1998).
- [18] F. Girelli, S. Liberati, and L. Sindoni, *Phys. Rev. D* **75**, 064015 (2007).
- [19] W. Siegel *in Fields*, *hep-th /9912205*, Chapter II.